Directed Networks with Spillovers∗
PASCAL BILLANDa, CHRISTOPHE BRAVARDa, SUDIPTA SARANGIB

aCREUSET, Jean Monnet University, Saint-Etienne, France.
email: pascal.billand@univ-st-etienne.fr
e-mail: christophe.bravard@univ-st-etienne.fr

bDepartment of Economics, Louisiana State University, Baton Rouge, LA 70803, USA. email: sarangi@lsu.edu

∗We would like to thank Hans Haller, Rob Gilles and Matt Jackson, for useful discussions.
Sudipta Sarangi acknowledges the support of NSF grant HSD-0527315 and the hospitality of CREUSET, Jean Monnet University.

Abstract

In this paper we examine the role played by different type of spillovers in network formation. More precisely, we study non-cooperative network formation in directed networks where the payoff of a player depends on the number of links she forms as well as, either on the total number of links formed by all other players, or the total number of links formed by her immediate neighbors. The first type of model is called the model with global spillovers while the second type is referred to as the local spillovers model. For both classes of games we investigate the existence of pure strategy Nash equilibria and characterize Nash networks under a number of different second order conditions on the payoff function.

JEL Classification: C72, D85
Key Words: one-way flow models, Nash networks, spillovers.
1 Introduction

In recent years there has been an upsurge of theoretical studies dealing with the formation of links between individual and the properties of networks that result from these links. These networks can be divided into two categories – directed networks and undirected networks, which give way to different theoretical frameworks. Following their introduction in the economics literature as the one-way flow model with homogeneous parameters by Bala and Goyal (2000, [1]), Galeotti (2006, [5]) explores the implications of heterogeneity in such networks. In this paper we add to this literature by exploring directed networks in the presence of spillovers. While networks allow for substantial spillovers, in a directed graph the spillovers are also directed and hence can have a significant impact on the equilibrium architectures.\footnote{We also refer the reader to Goyal and Joshi (2006, [6]) which is specifically devoted to spillovers of different types in undirected networks.}

In the typical one-way flow or directed network model, a person forming a link to another individual is able to access her information while the reverse is not possible. Such an act can range from accessing a webpage to citing someone’s work. In the context of an oligopolistic setup directed links may be viewed as the act of gathering information about a rival firm (see for instance Cohen and Levinthal (1990, [4]), and van Ryn (2000, [10])). Moreover, in many networks situations the payoff of a player does not only depend on the number of her links but also depends on the number of links of the other players. Such environments where the payoff of players also depends on the number of links established by other players are said to exhibit spillovers, and form the focus of our paper. More precisely, we examine how different types of spillovers generate asymmetries between players. In order to address this issue we study equilibrium networks, that is we investigate the issue of existence of Nash networks (or pure strategy Nash equilibrium) and characterize these networks.

We begin by considering situations where payoffs depend on the total number of links formed by the other players. This is called the model with the global spillovers property. As an example (see Section 3 below) we can think of situations where players have to decide on a level of effort to produce a pure public good and where link formation allows
players to obtain knowledge about the production of the good and increases the efficiency of players’ effort. In this setting we consider two separate restrictions on the payoff function and show that a Nash equilibrium always exists for each of these conditions. The first of these is a variation of discrete convexity called the strict smaller midpoint property. This condition merely imposes a monotonicity requirement on the marginal payoff function. The second restriction is derived from strategic substitutes (complements) property. We also characterize the Nash equilibria of the network formation game under each of these properties.

Next, we focus on a class of payoff functions satisfying the local spillovers property. Unlike the previous formulation, here the payoffs of a player depend on her own links and on the number of links formed by her immediate neighbors. As an example (see Section 4 below), consider situations where a firm \( i \) can improve its cost of production linking to firm \( j \) and obtaining information from it. Of course this link is more efficient when firm \( j \) has formed more links since it provides \( j \) access to more information. This is important for modeling benchmarking or the notion of imitating industry “best practices” popular in the strategic management literature. We simultaneously impose two properties on the payoff function to study local spillovers in directed models: concavity (convexity) in own links, and the property of strategic substitutes (complements). Under these properties, we obtain results on existence and characterization of Nash networks.

Although from a network perspective the local spillovers model is more interesting, the paper starts with the global spillovers framework. The setup in this case is simple and allows us to gain insights about Nash networks under two fairly simple properties which are imposed separately on the payoff function. The local spillover framework is more complex and the insights from the global spillovers case prove useful in this context and hence it is developed in the subsequent section.

Our paper is most closely related to the work of Goyal and Joshi (2006, [6]) who also examine spillovers in the context of networks. However, there are a number of differences. First, Goyal and Joshi (2006, [6]) deal with undirected networks while we focus on directed networks. The directed nature of the network can easily affect the equilibrium by restricting
the direction in which information flows. Second, our equilibrium notion differs from the one used by Goyal and Joshi (2006, [6]). Since they consider undirected networks, link formation requires mutual consent, while a player can unilaterally delete any subset of her links. Hence the notion of pairwise equilibrium used in their paper is appropriate for characterizing the predicted networks. By contrast, we deal with directed networks which do not require consent and do not allow for bi-directional flow of resources. Hence we use the notion of Nash equilibrium to characterize the equilibrium networks. It is noteworthy that the focus on situations where players form directed links instead of undirected links has an impact on the difficulty of the analysis. Third, as in Goyal and Joshi there exist situations where the formation of networks leads to asymmetries between ex ante identical players. However, these asymmetries are not the same since equilibrium networks exhibit different architectures in both models.

The rest of the paper is organized as follows. In section 2, we present the framework of our model. Section 3 focuses on networks with global spillovers. In section 4, we study networks with local spillovers and section 5 summarizes our results.

2 Model Setup

Link formation game. Let $N = \{1, \ldots, n\}$, with $n \geq 3$, be a finite set of players. Players make decisions about whether or not to form links with other players denoted by $g_{i,j} \in \{0, 1\}$. Here $g_{i,j} = 1$ implies that player $i$ forms a link with player $j$, while $g_{i,j} = 0$ means that player $i$ does not form such a link. Thus a strategy of player $i$ is given by $\mathbf{g}_i = (g_{i,j})_{j \in N \setminus \{i\}}$ while $\mathcal{G}_i$ denotes the set of all possible strategies of player $i$. A directed network $\mathbf{g} = \{(g_i)_{i \in N}\}$, which corresponds to a strategy profile is a formal description of the directed links that exist between the players. Denote by $\mathcal{G}$ the set of all such directed networks. We assume that the link $g_{i,j} = 1$ enables player $i$ to access $j$’s information, but not vice versa. In other words, the flow of resources in our model is one-way. Let $N_i(\mathbf{g}) = \{j \in N \mid g_{i,j} = 1\}$ be the set of players $j$ to whom $i$ is directly linked with, that is the set of immediate neighbors. Let $n_i(\mathbf{g})$ be the cardinal of $N_i(\mathbf{g})$. Denote by
\[ n_{-i}(g) = \sum_{j \neq i} n_j(g) \] the number of links in the network \( g \) excluding the links originating from player \( i \). Also we define two sets \( A = \{0, \ldots, n - 1\} \) and \( B = \{0, \ldots, (n - 1)^2\} \) purely for keeping track of players in the proofs.

**Network.** We now define the main network architectures that occur frequently in the paper. A network \( g \) is complete, and is denoted by \( g^c \), if for every pair of players \( i \) and \( j \), there is a link from \( i \) to \( j \) as well as a link from \( j \) to \( i \). The network \( g \) is empty if no player has formed a link and is denoted by \( g^e \). A network \( g \) is a \( k \)-all-or-nothing network if \( k \) players have established links with all other players, while \( n - k \) players have formed no links at all. Finally, given a network \( g \in \mathcal{G} \), let \( g_{-i} \) denote the network obtained when all of player \( i \)'s links are removed. Hence the network \( g \) can be written as \( g = g_{-i} \oplus g_i \), where \( \oplus \) indicates that \( g \) is formed by the union of links in \( g_i \) and \( g_{-i} \).

**Payoffs and Nash equilibrium.** In what follows, the amount of resources that a player obtains in the network depends on the number of links in the network. When a player’s payoff depends on the number of links she forms and the total number of links that other players form, we call this the network model with *global spillovers*. However, when a player’s payoffs depend on the number of direct links she forms as well as the total number of direct links established by her immediate neighbors, we call this the network model with *local spillovers*. Both models are defined more formally in subsequent sections of the paper. The payoff function of each player \( i \in N \) is given by \( u_i : \mathcal{G}_i \times \mathcal{G}_{-i} \to \mathbb{R} \), \( u_i : (g_i, g_{-i}) \mapsto u_i(g_i, g_{-i}) \). The strategy \( g_i \) is said to be a best response of player \( i \) to \( g_{-i} \) if we have:

\[ u_i(g_i, g_{-i}) \geq u_i(g'_i, g_{-i}), \]

for all \( g'_i \in \mathcal{G}_i \). The set of player \( i \)'s best responses to \( g_{-i} \) is denoted by \( \mathcal{BR}_i(g_{-i}) \). Finally, a network \( g = (g_1, \ldots, g_i, \ldots, g_n) \) is said to be a Nash network if \( g_i \in \mathcal{BR}_i(g_{-i}) \) for each \( i \in N \).
3 Networks with global spillovers

3.1 Definitions

In this section we consider network formation games in which the payoff of each player \(i\) can be expressed in terms of the number of links of player \(i\) and the aggregate number of links of the rest of the players. Hence the strategy of each player \(i\) can be summarized by the number of links, \(n_i(g)\), that player \(i\) forms and the strategies of the other players can be summarized by the number of links, \(n_{-i}(g)\), that players \(j \in N \setminus \{i\}\) form. Then the payoff function of player \(i\) can be written as:

\[
u_i(g, g_{-i}) = \phi(n_i(g), n_{-i}(g)).\]

We now present an example to motivate this payoff formulation.

Example 1 Cost reducing intelligence activities in oligopoly.\(^2\) Consider an homogeneous product Cournot oligopoly consisting of \(n\) ex ante symmetric firms which face the inverse demand function \(p = \alpha - \sum_{i \in N} q_i\), where \(\alpha > 0\). There are no fixed costs and firms have identical constant returns-to-scale cost functions. Forming links allows firms to collect information about other firms and such intelligence gathering lowers cost of production in a linear fashion: \(C_i(g) = \gamma_0 - \gamma n_i(g)\), where \(\gamma_0\) is a positive parameter representing a firm \(i\)'s marginal cost when it has no links. In this context \(g\) is the (directed) intelligence gathering architecture arising out of the link formation choices made by the firms. Given any network \(g\), the equilibrium output can be written as:

\[
q_i(g) = \frac{(\alpha - \gamma_0) + n\gamma n_i(g) - \gamma n_{-i}(g)}{(n + 1)}, i \in N.
\]

The Cournot profits for firm \(i \in N\) are given by

\[
\phi(n_i(g), n_{-i}(g)) = (q_i(g))^2 - fn_i(g),
\]

where \(f\) is the cost of establishing a link.

\(^2\)This model is taken from Billand and Bravard (2004, [2]).
To analyze the model further we now introduce two properties of the payoff function with global spillovers (see also Goyal and Joshi (2006, [6]), for two similar properties).

**Definition 1** Suppose $n_{-i}(g') > n_{-i}(g)$. The payoff function satisfies the strategic complements (substitutes) property, if $\phi(n_i(g) + 1, n_{-i}(g')) - \phi(n_i(g), n_{-i}(g')) > (\phi(n_i(g) + 1, n_{-i}(g)) - \phi(n_i(g), n_{-i}(g)))$.

In other words, we say that a payoff function satisfies the strategic complements (substitutes) property when the marginal payoff function of a player $i$ from establishing links is increasing (decreasing) in the number of links formed by the other players. The idea behind the strategic complements property (SCP) is that the value of information possessed by a player increases as other players form more links, while the strategic substitutes property (SSP) says that it decreases. Hence the SSP property is more useful for instance when the exclusivity of the information matters and SCP is appropriate for situations where awareness and popularity matters, i.e, marketing or when word-of-communication is crucial. Finally, note that these properties can be used even if the arguments belong to a discrete space.

While the SCP (SSP) property deals with the marginal payoff of player $i$ based on the links formed by other players, the following property takes the player’s own links into account.

**Definition 2** The payoff function is convex (concave) in own links if the marginal payoff, $\phi(n_i(g) + 1, n_{-i}(g)) - \phi(n_i(g), n_{-i}(g))$, is strictly increasing (decreasing) in $n_i(g)$.

However, in this section, we use a modified version of this property. Given the nature of global spillovers it is possible to use a weaker property called the strict smaller midpoint property. This property is inspired by Ui (2005, [9]).\(^3\) Let $|x| = \max\{-x, x\}$.

**Definition 3** We say that a function $f : A \to \mathbb{R}$ satisfies the strict smaller midpoint property if, for any $x, y, z \in A$, with $|x - y| = 2$, and $|z - x| = |z - y| = 1$, there exists

\(^3\)Ui (2005, [9]) deals with discrete concavity and provides a more general definition of the larger midpoint property.
\( t \in (0, 1), \) such that,

\[
f(z) < tf(x) + (1 - t)f(y).
\]

First, we discuss the relationship between the strict smaller midpoint property and the convexity of Definition 2. It is clear that if \( \phi \) is convex in own links, then \( \phi \) satisfies the strict smaller midpoint property in \( n_i(g) \) in its first argument. Indeed, if \( \phi(n_i(g) + 1, n_{-i}(g)) - \phi(n_i(g), n_{-i}(g)) \) is increasing, then \( \phi(n_i(g) + 1, n_{-i}(g)) - \phi(n_i(g), n_{-i}(g)) > \phi(n_i(g), n_{-i}(g)) - \phi(n_i(g) - 1, n_{-i}(g)) \). It follows that \( (1/2)(\phi(n_i(g) + 1, n_{-i}(g)) - \phi(n_i(g) - 1, n_{-i}(g))) > \phi(n_i(g), n_{-i}(g)) \) and \( \phi \) satisfies the strict smaller midpoint property. However, let \( \phi_i(n_i(g), n_{-i}(g)) = (n_i(g) - 2)^3 + n_{-i}(g) \) be the payoff function of each player \( i \). It is obvious that \( \phi \) is concave (not convex) and satisfies the strict smaller midpoint property.\(^4\)

Thus, the strict smaller midpoint property is a weaker property than the notion convexity of defined above. Moreover, in our context, if a function is monotonic, then it satisfies the strict smaller midpoint property.

Second, we discuss the relationship between the strict smaller midpoint property and the discrete convexity. Note that, in defining the strict smaller midpoint property, we postulate that the midpoint of \( x, y \in A \) is \( z \in A \). We say that a function \( f : A \to \mathbb{R} \) is discretely convex if for all \( x, y \in A \) and for all \( \alpha \in (0, 1) \)

\[
\alpha f(x) + (1 - \alpha)f(y) > \min_{\xi \in N(z)} f(\xi),
\]

where \( N(z) = \{ \xi \in A : |\xi - z| < 1 \} \), \( z = \alpha x + (1 - \alpha)y \).\(^5\) It is easy to check that in our context if \( f \) is discretely convex, then it satisfies the smaller midpoint property (it is enough to set \( \alpha = 1/2 \) and \( y = x + 2 \)).

\textbf{Example 2} \textit{Provision of a pure public good.}\(^6\) There are \( n \) players, each of whom decides on the level of output, \( x_i \), to produce a pure public good. Let \( x_{-i} \) be the vector of output

\(^4\)Likewise, there exist payoff functions which are neither convex, nor concave which satisfy the strict smaller midpoint property. Consider for instance \( \phi_i(n_i(g), n_{-i}(g)) = (n_i(g) - 2)^3 + n_{-i}(g) \).

\(^5\)This definition is a slight variation of the definition given by Yuceer ([11], 2002) in our context.

\(^6\)This is qualitatively similar to example 4.1 in Goyal and Joshi (2006, [6]). It has been modified to take into account the directed nature of the network and to ensure that the utility function satisfies the strict smaller midpoint property.
levels of all players except \( i \). Given each player’s output, the utility of player \( i \) is:

\[
\phi(x_i, x_{-i}) = x_i + \sum_{j \in N \setminus \{i\}} x_j.
\]

A link formed by player \( i \) with \( j \) allows the former to obtain knowledge about the production of the public good and increases the efficiency of player \( i \)’s effort for producing output \( x_i \). Therefore the cost of producing \( x_i \), gross of the costs of links, decreases with the number of links formed by player \( i \). More specifically, we assume that the cost of producing \( x_i \), gross of the costs of links, in a network \( g \) is given by:

\[
p_i(x_i, g) = \frac{x_i^2}{2\eta_i(g)},
\]

where we set \( \eta_i(g) = n_i(g) + 1 \). Given a network \( g \), each player \( i \) will choose output to maximize utility net of production costs. This yields an optimal output of \( x_i(g) = \eta_i(g) \). Therefore, the reduced form gross payoff of player \( i \), in a network \( g \), is:

\[
\frac{1}{2} \eta_i(g) + \sum_{j \in N \setminus \{i\}} \eta_j(g).
\]

We assume that the cost of forming links is \( \gamma \sqrt{n_i(g)} \), \( \gamma > 0 \), so the payoff of player \( i \) is:

\[
\phi(n_i(g), n_{-i}(g)) = n - \frac{1}{2} n_i(g) + \sum_{j \in N \setminus \{i\}} n_j(g) - \gamma \sqrt{n_i(g)}.
\]

It is clear that this payoff function satisfies the strict smaller midpoint property.

### 3.2 Existence and characterization of Nash networks

We say that two networks \( g \) and \( g' \) are adjacent if there is a unique player \( i \in N \) such that \( g_{i,j} \neq g'_{i,j} \) for at least one player \( j \neq i \) and if for all players \( k \neq i \), \( g_{k,j} = g'_{k,j} \), for all \( j \in N \). An improving path is a sequence of adjacent networks that results when players form or sever links based on strict payoff improvement the new network offers over the current network. Each network in the sequence differs from the previous one by the link(s) formed by one unique player. Formally, an improving path from a network \( g \) to a network \( g' \) is a finite sequence of networks \( g^1, \ldots, g^k \), with \( g^1 = g \) and \( g^k = g' \), such that the two following conditions are satisfied:
1. $g^\ell$ and $g^{\ell+1}$, are adjacent networks;

2. for the unique player, say player $i$, who has changed her links, we have $g_i^{\ell+1} \in BR_i(g_{\ell-i})$ and $g_i^{\ell} \notin BR_i(g_{\ell-i})$, that is $g_i^{\ell+1}$ is a network where $i$ plays a best response while $g_i^{\ell}$ is a network where $i$ does not play a best response.

Moreover, if $g^1 = g^k$, then the improving path is called an improving cycle. Clearly, a network $g$ is a Nash network if and only if it has no improving path emanating from it.

Our first result proves existence and provides a complete characterization of equilibrium architectures in games with positive spillovers across own links, that is when the payoff function satisfies strict smaller midpoint property. To simplify the notations, let $g_i = (0, 0, \ldots, 0) = 0$ and $g_i = (1, 1, \ldots, 1) = n - 1$, be the two polar strategies for player $i \in N$.

**Lemma 1** Suppose that the payoff function satisfies the strict smaller midpoint property. Then, the best response of each player $i \in N$ is either $0$ or $n - 1$.

**Proof** See appendix \qed

**Proposition 1** Suppose that the payoff function satisfies the strict smaller midpoint property. Then the one-way flow model with global spillovers always contains a Nash network in pure strategies. Moreover, the Nash networks are $k$-all-or-nothing networks.

**Proof** We prove successively both parts of the proposition.

1. To prove the first part we start from the empty network $g^e$, and show that we can reach a Nash network, i.e. there is no improving cycle originating from the empty network. Clearly if there is no improving path from $g^e$, we are done. Otherwise, there exists a player, say $i_1$, such that $0 \notin BR_{i_1}(g^{e}_{-i_1})$. Hence, by Lemma 1, we have, $BR_{i_1}(g_{i_1}^{e-i_1}) = n - 1$. Let $g^1$ be the network in which no player has formed links except player $i_1$ who has formed $n - 1$ links. Either $g^1$ is a Nash network and we are done, or there is a player say $i_2$ such that $0 \notin BR_{i_2}(g^{1}_{-i_2})$. In the latter case, by Lemma 1, we have $BR_{i_2}(g^{1}_{i_2}) = n - 1$. Let $g^2$ be the network in which no player has formed links except player $i_2$ who has formed $n - 1$ links. Either $g^2$ is a Nash network and we are done, or there is a player say $i_3$ such that $0 \notin BR_{i_3}(g^{2}_{-i_3})$. In the latter case, by Lemma 1, we have $BR_{i_3}(g^{2}_{i_3}) = n - 1$. Let $g^3$ be the network in which no player has formed links except player $i_3$ who has formed $n - 1$ links. Either $g^3$ is a Nash network and we are done, or there is a player say $i_4$ such that $0 \notin BR_{i_4}(g^{3}_{-i_4})$. In the latter case, by Lemma 1, we have $BR_{i_4}(g^{3}_{i_4}) = n - 1$. Let $g^4$ be the network in which no player has formed links except player $i_4$ who has formed $n - 1$ links. This process can be continued until we reach a Nash network.
links except players $i_1$ and $i_2$ who have formed $n-1$ links. We observe that player $i_1$ has no incentive to modify her strategy in $g^2$. Indeed, we have $BR_i = BR_j$ for all $i \in N, j \in N$, and by construction $g^2_{-i_1} = g^2_{-i_2}$. Therefore, if $BR_{i_1}(g^1_{-i_2}) = n - 1$, then $BR_{i_1}(g^2_{-i_1}) = n - 1$. More generally, we define $g^k$ the network in which no player has formed links except players $i_1, i_2, \ldots, i_k$ who have formed $n-1$ links and $BR_i(g^k_{-i}) = n - 1$ for all $i \in \{i_1, \ldots, i_k\}$. Either $g^k$ is a Nash network and we are done, or there exists a player, say $i_{k+1}$, such that $0 \notin BR_{i_{k+1}}(g^k_{-i_{k+1}})$. By Lemma 1, $BR_{i_{k+1}}(g^k_{-i_{k+1}}) = n - 1$. Let $g^{k+1}$ be the network in which no player has formed links except players $i_\ell$, with $\ell \in \{1, \ldots, k+1\}$ who has formed $n-1$ links. We observe that players $i_\ell$ have no incentive to modify their strategy in $g^{k+1}$ since $BR_i(\cdot) = BR_j(\cdot)$ for all $j, i \in N$ and $g^{k+1}_{-i} = g^{k+1}_{-i_\ell}$ for all $i \in \{i_1, \ldots, i_k\}$. Hence, there does not exist an improving cycle starting from $g^k$ and, since the set of players $N$ is finite, a Nash network in pure strategies always exists.

2. The characterization of Nash networks follows immediately from Lemma 1: in a Nash network each player forms either 0 or $n-1$ links.

\[\square\]

From the above proposition it is clear that if the payoff function satisfies the strict smaller midpoint property, then there exist situations where players are in asymmetric positions. In other words, \textit{ex ante} identical players obtain different payoffs. Recall that in Goyal and Joshi (2006 [6], Proposition 3.1, pg.327) equilibrium networks are either empty, complete, or have the dominant group architecture.\textsuperscript{7} The difference between the equilibrium architectures in the two papers can be explained as follows: while isolated players can prevent players in the dominant group from establishing a link with them in the Goyal and Joshi framework, this is not possible in our model due to the noncooperative nature of links.

Moreover, observe that in our framework asymmetries between players, when there exist, are stronger. Indeed, recall that in $k$-all-or-nothing networks there is a set of players who

\textsuperscript{7}In a dominant group architecture, there are two groups of players: one in which each player is connected with all other members of the group and a second in which players are isolated.
obtain resources from all the others as well as players who obtain no resources from any other player.

**Example 2 (revisited).** Since the payoff function in Example 2 satisfies the strict smaller midpoint property, a Nash network always exists in this game. Moreover, by Proposition 1 these Nash networks are $k$-all-or-nothing networks.

We now use the notion of SCP (or SSP) to study the existence of Nash networks and characterize them. We begin with a necessary definition. A **symmetric network** $g$ is a network in which all players form the same number of links, i.e. for all $i \in N$ and $j \in N$, we have $n_i(g) = n_j(g)^8$.

**Proposition 2** In a one-way flow model with global spillovers and SSP (or SCP), a Nash network in pure strategies always exists.

1. Suppose the payoff function satisfies SCP. Every Nash network is a symmetric network. Moreover, there exists a function $\phi$ such that any symmetric network $g$ is Nash.

2. Suppose the payoff function satisfies SSP. All networks can be supported as Nash equilibria.

**Proof** See appendix. □

Observe that when the payoff function satisfies SCP, all players are in symmetric positions in an equilibrium, and hence obtain the same payoffs. By contrast, if the payoff function satisfies SSP, then there exist situations where players are in asymmetric positions in an equilibrium network and therefore do not obtain the same payoffs in equilibrium even though they are *ex ante* identical.

Our results differ from those of Goyal and Joshi (2006, [6]) in two ways. In the Goyal and Joshi framework it is not possible to obtain existence results when the payoff function satisfies the strategic substitutes property (Proposition 3.3, pg.329) while in our case Nash equilibria always exist. Second, we do not obtain the same equilibrium networks. Indeed, 

---

8In graph theory such a network is also called a regular graph.
in contrast to Goyal and Joshi (Proposition 3.2, pg.328), asymmetric networks cannot be equilibrium networks when payoff function satisfies SCP.

Next, we explain what happens under global spillovers when we impose SCP (or SSP) and the strict smaller midpoint properties simultaneously on the payoff function.

**Corollary 1** Suppose $\phi$ satisfies the strict smaller midpoint property.

1. Suppose $\phi$ satisfies SCP. Then, a Nash network is either the empty network or the complete network.

2. Suppose $\phi$ satisfies SSP. (a) The complete network is the unique Nash network if and only if $\phi(n-1,(n-1)^2) > \phi(0,(n-1)^2)$. (b) The empty network is a Nash network if $\phi(0,0) > \phi(n-1,0)$.

**Proof** We prove both parts successively.

1. The result is obvious from proposition 1 and proposition 2.1.

2. Since 2(b) is straightforward, we only prove the first part. First, it is obvious that if the complete network is a Nash network, then $\phi(n-1,(n-1)^2) > \phi(0,(n-1)^2)$. Second, we show that if $\phi(n-1,(n-1)^2) > \phi(0,(n-1)^2)$ then the complete network is the unique Nash network. Indeed, assume that there is a non complete network $g^*$ which is Nash. We have $\phi(n_i(g^*),n_{-i}(g^*)) \geq \phi(n_i(g),n_{-i}(g^*))$, for all $g \in G$ and for all $i \in N$. By SSP we have:

$$\phi(n-1,(n-1)^2) > \phi(0,(n-1)^2) \Rightarrow \phi(n-1,n_{-i}(g)) > \phi(0,n_{-i}(g)),$$

for all $n_{-i}(g) \in \{0,\ldots,(n-1)^2-1\}$. By Lemma 1, and the strict smaller midpoint property, we have:

$$\phi(n-1,n_{-i}(g)) > \phi(0,n_{-i}(g)) \Rightarrow \phi(n-1,n_{-i}(g)) > \phi(n_i(g),n_{-i}(g)),$$

for all $n_i(g) \in \{0,\ldots,n-2\}$. This tells us that each player $i$ always has an incentive to form $n-1$ links in $g^*$. Hence, $g^*$ cannot be a Nash network giving us the necessary contradiction.
Example 1 revisited. It is clear that the payoff function satisfies SSP and the strict smaller midpoint property. Hence using Proposition 1 we conclude that Nash networks are \( k \)-all-or-nothing networks. That is, there are at most two groups of firms. Moreover, by Corollary 1, we know that the complete network is the unique Nash network if and only if \( \phi(n-1, (n-1)^2) > \phi(0, (n-1)^2) \) and the empty network is a Nash network if \( \phi(0, 0) > \phi(n-1, 0) \).

4 Networks with local spillovers

In this section, we consider a class of games where the aggregate payoffs of player \( i \) can be written as:

\[
  u_i(g_i, g_{-i}) = \Psi_1(n_i(g)) + \sum_{j \in N_i(g)} \Psi_2(n_j(g)) + \sum_{j \notin N_i(g)} \Psi_3(n_j(g)),
\]

where \( \Psi_k, k = 1, 2, 3 \) are functions. In other words, in this class of games, the payoff of player \( i \) depends on the identity of the players inside and outside her immediate neighborhood. Without loss of generality we suppose that player \( i \) obtains a null payoff when she forms no link. Note that in this framework, spillovers capture the benefits or costs transmitted across links when a player forms a link and gains access to the partner’s links. Here the marginal return to player \( i \) from forming a link with \( j \) depends on both the number of links of player \( i \) and the number of links of her immediate neighbor \( j \).

\[
  u_i(g_i \oplus g_{i,j}, g_{-i}) - u_i(g_i, g_{-i}) = \psi_1(n_i(g) + 1) - \psi_1(n_i(g)) + \psi_2(n_j(g) + 1) - \psi_3(n_j(g)) = \psi(n_i(g) + 1, n_j(g))
\]

To simplify the analysis we assume that \( \psi(n_i(g) + 1, n_j(g)) - \psi(n_i(g), n_j(g)) \neq 0 \) for all \((n_i(g), n_j(g)) \in \{0, \ldots, n-2\} \times \{0, \ldots, n-1\}\). Observe that the two types of externality effects now operate through the own links of the players, and through the links of the potential partner. Such networks where the spillovers are local often arise in practice. Consider

\[\text{[Note: This payoff function was introduced by Goyal and Joshi (2006, [6]).]}\]
for instance the standard pyramid marketing scheme. As player $i$’s immediate neighbors establish more direct links, players $i$’s payoffs increase. On the other hand consider a meta search engine that provides links to a number of sites. As each of those sites provides links to each other’s sites, the payoffs of the meta search engine will fall. Now we present a formal example relating to the practice of benchmarking among firms.

Example 3 Learning from others. Our learning from others example specifically focuses on the strategic management notion of “benchmarking” (Camp, 1989, [3]). Consider $n$ firms. For simplicity assume that each firm $i$ faces unit price and aims to produce a given quantity $\bar{Q}_i$ at the lowest total production cost. We assume that each firm $i$ can observe business processes of its competitors through the links that exist in the network in order to identify and imitate best practices. The production cost of each firm $i$ is given by:

$$C_i = \alpha_i + \beta n_i(g) + \gamma n_i(g)^2 + \sum_{j \in N_i(g)} (\alpha'_j + \beta' n_j(g)),$$

with $\alpha_i, \alpha'_j > 0$ for all $i \in N$, $j \in N$ and $\beta < 0$, $\beta' < 0$, $\gamma \in \mathbb{R}$. In order to ensure that the cost is positive, we will assume that $\alpha_i + (n-1)\beta + \gamma(n-1)^2 + \alpha'_j + \beta'(n-1)^2 > 0$ if $\gamma < 0$, and $\alpha_i + (n-1)\beta + \alpha'_j + \beta'(n-1)^2 > 0$ if $\gamma \geq 0$. We can now write the payoff function as:

$$u_i(g_i, g_{-i}) = \bar{Q}_i - \left( \alpha_i + \beta n_i(g) + \gamma n_i(g)^2 + \sum_{j \in N_i(g)} (\alpha'_j + \beta' n_j(g)) \right).$$

Let $g'_i$ and $g_i$ be such that $g'_{i,k} = g_{i,k}$ for all players $k \in N \setminus \{j\}$ and $g'_{i,j} = 1 \neq 0 = g_{i,j}$. Then the marginal payoff of firm $i$, if it forms a link with firm $j$ is:

$$u_i(g'_i, g_{-i}) - u_i(g_i, g_{-i}) = -\beta - \gamma(2n_i(g) + 1) - \alpha' - \beta' n_j(g),$$

The returns from a link with a firm $j$ depends on the efficiency of firm $j$’s processes and this efficiency depends on how many links firm $j$ has established with other firms. This measure how much $j$ learns from the others. More precisely, the larger the number of links firm $j$ has, the higher is the payoff of forming a link with this firm ($\beta' < 0$). Moreover, the returns from a link with a firm $j$ for firm $i$ depends on the number of links formed by $i$.

We can have $\gamma \geq 0$ or $\gamma < 0$ according to the context.
Next we define some properties of the payoff function before proceeding to deal with the existence and characterization of Nash networks.

**Definition 4** The payoff function is convex (concave) if \( \psi_1(k+1) - \psi_1(k) \) is strictly increasing (decreasing).

**Definition 5** The payoff function satisfies the strategic substitutes (complements) property if \( \psi_2(k+2) - \psi_3(k+1) < (>) \psi_2(k+1) - \psi_3(k) \).

**Proposition 3** Suppose that payoff function satisfies equation (1) and strategic complements property (SCP).

1. If the payoff function is convex in own links, then a Nash network in pure strategies always exists. Further, a Nash network is either the empty network, or the complete network.

2. If the payoff function is concave in own links, then a Nash network in pure strategies always exists. Further such a Nash network is symmetric.

**Proof** We prove successively both parts of the proposition.

1. To prove the existence of Nash networks, we start with the empty network \( g^e \). Either \( g^e \) is Nash and we are done, or there is a player, say \( i_1 \), who has an incentive to form links. In that case, there is \( i \in \{1, \ldots, n-1\} \) such that \( \sum_{k=1}^i \psi(k,0) > 0 \). Clearly the best response for player \( i_1 \) is to play strategy \( n-1 \). Indeed, by convexity, we have: \( \psi(i,0) > \psi(i-1,0) \), so \( \sum_{k=1}^{i-1} \psi(k,0) \geq \sum_{k=1}^i \psi(k,0) > 0 \), for all \( i \in \{1, \ldots, n-1\} \). Denote the resulting network as \( g^1 \). By the strategic complements property, it is clear that players \( j \neq i_1 \) are not playing a best response in \( g^1 \) since \( \psi(n-1,n-1) + \sum_{k=1}^{n-2} \psi(k,0) > \sum_{k=1}^{n-1} \psi(k,0) > 0 \). Player \( i_2 \neq i_1 \) is chosen to play a best response in \( g^1 \). It is clear that \( n-1 \in \mathcal{BR}_{i_2}(g^1_{-i_2}) \). These arguments can be repeated to show that all players will form \( n-1 \) links and players who have already form \( n-1 \) links will not deviate from this strategy. So, the complete network will be Nash. We can use the same argument to prove that a Nash network is either the empty network, or the complete network.
2. To prove the existence of Nash networks, again we start with the empty network $g^e$. Either $g^e$ is Nash and we are done, or there is a player, say $i_1$, who has an incentive to form links.

**Step 1.** Order the players in some pre-specified manner, 1, 2, ..., $n$ and assume that they are given in turn the option to revise their actions concerning their links. We begin with player 1. Note that either there is $k \in \{1, \ldots, n - 2\}$ such that $\psi(k, 0) > 0$ and $\psi(k + 1, 0) < 0$, or $\psi(n - 1, 0) > 0$. Otherwise the empty network is Nash. We assume that player 1 forms links with players $n$, $n - 1$, ..., $n - k + 1$ and we obtain the network $g'$. It is clear that players $j \neq 1$ are not playing a best response in $g'$ since $\psi(k, k) + \sum_{i=1}^{k-1} \psi(i, 0) > \sum_{i=1}^{k} \psi(i, 0) > 0$. We now allow player 2 to play a best response in $g'$. Player 2 must change her strategy and form a link with player 1 and at least $k - 1$ links with players $j \in N \setminus \{1, 2\}$. Indeed, player 2 cannot form fewer than $k$ links since $\psi(k, 0) > 0$. We assume that player 2 forms links with players 1, $n$, $n - 1$, ... We pursue this process till player $n$. At the end of this process, we obtain a network $g^1$. It is clear that players $j \neq 1$ are not playing a best response in $g^1$ since $\psi(k, k) + \sum_{i=1}^{k-1} \psi(i, 0) > \sum_{i=1}^{k} \psi(i, 0) > 0$. We now allow player 2 to play a best response in $g^1$. Player 2 must change her strategy and form a link with player 1 and at least $k - 1$ links with players $j \in N \setminus \{1, 2\}$. Indeed, player 2 cannot form fewer than $k$ links since $\psi(k, 0) > 0$. We assume that player 2 forms links with players 1, $n$, $n - 1$, ... We pursue the process till player $n$. At the end of this process, we obtain a network $g^1$ such that $\sum_{j\in N} n_j(g^1) > \sum_{j\in N} n_j(g^e) = 0$. It is clear, by the strategic complements property that $n_k(g^1) \geq n_{k-1}(g^1)$.

**Step 2.** We start with network $g^1$ and again we let players 1, 2, ..., $n$ choose a best response successively. In network $g^1$, player 1 has no incentive to decrease her number of links. Indeed, we have $\psi(k, n_{n-k+1}(g^1)) > \psi(k, 0) > 0$. Moreover, by construction of the process, if player 1 preserves her number of links, then she does not modify her strategy. Indeed, since $n_k(g^1) \geq n_{k-1}(g^1)$, player 1 has no incentive to replace a link with a player $j \in \{n - k + 1, \ldots, n\}$ by a link with a player $j \in \{2, \ldots, n - k\}$ because of the strategic complements property and the construction of the process. If player 1 has an incentive to add some links, then she chooses to form links with players $n - k$, $n - k - 1$, .... We pursue the process till player $n$ and we obtain the network $g^2$. Now there are two possibilities: either $\sum_{j\in N} n_j(g^1) = \sum_{j\in N} n_j(g^2)$ or $\sum_{j\in N} n_j(g^1) < \sum_{j\in N} n_j(g^2)$. In the first case, $g^1 = g^2$ and $g^1$ is Nash. In the second case, some players have added some links and in that case we reiterate the process.

Since the number of links that players can form in a network is bounded by $n(n - 1)$,
the process converges toward a Nash network.

We now show that a Nash network is symmetric. Suppose that the network $g$ is not symmetric. Then, there exist players $i$ and $j$ such that $n_i(g) > n_j(g)$. It follows that there exists a player $k \in N$ such that $k \in N_i(g)$ and $k \not\in N_j(g)$. There are two possibilities: either $k \neq j$ or $k = j$. If $k \neq j$, then we have $\psi(n_i(g), n_k(g)) > 0$. By the concavity property and SCP, we have: $\psi(n_j(g) + 1, n_k(g)) \geq \psi(n_i(g), n_k(g)) > 0$ and player $j$ is not playing a best response in $g$, which is a contradiction. If $k = j$, then $\psi(n_j(g) + 1, n_i(g)) > \psi(n_i(g), n_j(g)) > 0$ and player $j$ is not playing a best response in $g$, which is again a contradiction.

\[\square\]

Observe that this result is very close to the result obtained when spillovers are global: all players are in a symmetric position and obtain the same payoff. Further note that if the payoff function satisfies the SCP property and is convex in own links, we obtain results which are simpler than the results obtained by Goyal and Joshi (2006, [6]) since in their framework both symmetric and asymmetric networks can be stable. This difference is mainly due to the fact that we have directed networks where link formation does not involve a consent requirement. This lack of consent requirement does confer benefits in some cases. We are able to provide results about existence and characterization of Nash networks when the payoff function satisfies the strategic complements property and is concave in own links without requiring any additional conditions.

**Example 3 revisited.** The payoff function in Example 3 always satisfies strategic complement properties. Moreover, this function is either concave ($\gamma \geq 0$), or convex ($\gamma \leq 0$). In both cases, by Proposition 3 there always exists a Nash network. Moreover, while in the former case Nash networks are symmetric, in the latter case Nash networks are empty or complete.

The next proposition concerns situations with local spillovers and where the payoff function satisfies strategic substitutes property. Note that in this context Goyal and Joshi (2006, [6], Propositions 4.2, pg.337) need additional conditions on the monotonicity of the
payoff function to characterize equilibrium networks.

We now provide two definitions useful in the following proposition. Let \( N^k(g) = \{ i \in N | |N_i(g)| = k \} \) be the set of players who have formed \( k \) links in \( g \) and let \( \mathcal{N}(g) = \{ N^k(g) | N^k(g) \neq \emptyset \} \) be the set of all \( N^k(g) \) which are non empty.

**Proposition 4** Suppose that payoff function satisfies equation (1) and strategic substitutes property (SSP). Suppose the payoff function is convex or concave in own links. Then a Nash network in pure strategies always exists. Moreover,

1. if the payoff function is convex in own links, then a Nash network \( g \) is such that for all \( i \in N \) and \( j \in N \), we have either \( N_i(g) \subseteq N_j(g) \) or \( N_i(g) \supseteq N_j(g) \);

2. if the payoff function is concave in own links, then a Nash network \( g \) is such that \( |\mathcal{N}(g)| \leq 2 \).

**Proof** See appendix. \(\Box\)

For the sake of illustration, we give a network which satisfies the property listed in the Proposition 4.1.

![Network](image)

Figure 1: A network \( g \) which satisfies \( N_i(g) \subset N_j(g) \).

Unlike the previous case, when the payoff function satisfies SSP, there exist situations where \( ex \ ante \) identical players are in asymmetric situations and obtain different payoffs in equilibrium. Furthermore, the equilibrium network \( g \) shown in Figure 1 which we call a “hierarchical star” has not been seen in the literature previously. This kind of network is quite asymmetric since each player \( i \) in the network obtains an amount of resources that are different from those obtained by player \( j, j \neq i \). In other words, again there exist situations where \( ex \ ante \) identical players are in different positions in equilibrium networks.
5 Conclusion

This paper examines directed networks where the payoff of a player depends on the number of links she forms as well as either on the total number of links formed by all other players, or the total number of links formed by her immediate neighbors. The first type of model is called the model with global spillovers. We investigate the existence of pure strategy Nash equilibria and characterize the equilibrium networks under a discrete convexity property as well as under SCP or SSP. The discrete convexity property, used in this paper, leads to \( k \)-all-or-nothing networks in equilibrium. Under SCP we find that equilibrium networks are symmetric, while under SSP all networks can be supported as Nash equilibria.

From a network perspective the second class of models called local spillovers are more interesting. These are analyzed under the SSP (SCP) property along with concavity or convexity of the payoff function. Again we characterize the equilibrium networks and prove existence of pure strategy Nash equilibria. Under SCP, we find that equilibrium networks are always symmetric (regardless of whether the payoff function is concave or convex). This is an important insight that carries over from the global spillovers model. Under SSP we find equilibrium architectures that have not been identified before in the literature. Indeed SSP along with convexity of payoffs in own links gives us a hierarchy of star networks while under concavity we get directed networks with agents selecting at most two types of roles in link formation.

We find that there are mainly two differences between the Goyal and Joshi’s framework and our model. First, the formation of a link needs consent in Goyal and Joshi but not in our model. Second, an additional link implies that both players involved obtain additional resources in the Goyal and Joshi framework while an additional link implies that only one player obtains more resources in our setting. Although the answer is not unequivocal, these do lead to different results. For instance, the architectures found in Proposition 1 can be attributed to the fact that links are non-cooperative. However, the existence results obtained in Proposition 2 are due to the fact that the links are directed, and an additional link does increase payoffs of both players.
6 Appendix

6.1 Proof of Lemma 1

In order to prove this lemma we need the following two lemmas which are stated without proofs.\(^\text{10}\) Recall also that \(A = \{0, \ldots, n - 1\}\) and \(B = \{0, \ldots, (n - 1)^2\}\).

**Lemma 2** Let the function \(\phi: A \times B \to \mathbb{R}\) satisfy the strict smaller midpoint property in its first argument. If \(\phi(n - 1, w) \leq \phi(n - 2, w)\) for all \(w \in B\), then \(\phi(z, w) > \phi(z + 1, w)\) for all \(z \in \{0, \ldots, n - 3\}\).

**Lemma 3** Let the function \(\phi: A \times B \to \mathbb{R}\) satisfy the strict smaller midpoint property in its first argument. Then, for any \(w \in B\), we have

\[
\max\{\phi(0, w), \phi(n - 1, w)\} > \phi(z, w), \forall z \in \{1, \ldots, n - 2\}.
\]

**Proof of Lemma 1** By Lemma 2 we know that if \(\phi(n - 1, w) \leq \phi(n - 2, w)\), then \(\phi(z, w) < \phi(z - 1, w)\), for all \(z \in \{1, \ldots, n - 1\}\). Then, we have \(\phi(0, w) > \phi(z, w)\) for all \(z \in \{1, \ldots, n - 1\}\). Also, assume that \(\phi(n - 1, w) > \phi(n - 2, w)\). There are now two cases.

1. Suppose \(\phi(0, w) < \phi(1, w)\). Then by Lemma 3, \(\phi(n - 1, w) > \phi(z, w)\) for all \(z \in \{1, \ldots, n - 2\}\).

2. Suppose \(\phi(0, w) \geq \phi(1, w)\) for all \(w \in B\). Then, we show that there exists a unique \(d \in \{2, \ldots, n - 2\}\) such that \(\phi(d, w) < \phi(d - 1, w)\), and \(\phi(d, w) < \phi(d + 1, w)\).

- If such a \(d\) does not exist, then we know that \(\phi(\cdot, w)\) is decreasing in its first argument and we have a contradiction since \(\phi(n - 1, w) > \phi(n - 2, w)\).

- Suppose there exist \(d\) and \(d', d \neq d'\), such that \(\phi(d, w) < \phi(d - 1, w)\), \(\phi(d, w) < \phi(d + 1, w)\) and \(\phi(d', w) < \phi(d' - 1, w)\), \(\phi(d', w) < \phi(d' + 1, w)\). Without loss of generality let \(d' > d\). Since \(\phi(d, w) < \phi(d + 1, w)\), we have \(\phi(d + 1, w) < \phi(d + 2, w)\) and by induction \(\phi(d + k, w) < \phi(d + k + 1, w)\) for all \(k \in \{1, \ldots, n - d - 2\}\) and \(w \in B\). Hence, there does not exist \(d' \in \{d + 2, \ldots, n - 2\}\) such that \(\phi(d', w) < \phi(d' - 1, w)\), \(\phi(d', w) < \phi(d' + 1, w)\) which yields a contradiction.

\(^{10}\)These proofs are straightforward and can be found in the working paper version.
Hence, for all \( w \in B \) and for all \( z \in \{1, \ldots, d\} \), we have \( \phi(0, w) > \phi(z, w) \), and for all \( z \in \{d, \ldots, n-2\} \), we have \( \phi(n-1, w) > \phi(z, w) \). This gives us the desired conclusion.

\[ \square \]

### 6.2 Proof of Proposition 2

Here we show that SCP and SSP are sufficient conditions to obtain the existence of Nash networks. We use two results due to Zhou (1994, [12]) and Kukushkin (1994, [7]). We first need the following definitions to present these results.

**Definition 6** (Topkis 1998, [8]) If \( X \) and \( T \) are sets, \( S_t \) is a subset of \( X \) for each \( t \) in \( T \), and \( x_t \) is in \( S_t \), then the function \( x_t \) from \( T \) into \( X \) is a selection from \( S_t \). If \( X \) and \( T \) are partially ordered sets, \( S_t \) is a subset of \( X \) for each \( t \) in \( T \), and \( x_t \) is a selection from \( S_t \) that is an increasing (decreasing) function of \( t \) from \( T \) into \( X \), then \( x_t \) is an increasing (decreasing) selection.

**Definition 7** Zhou (1994, [12]) A correspondence \( F \) is ascending if, for any \( x \geq y \), any \( s \in F(x) \), and any \( t \in F(y) \), it is true that \( \max\{s, t\} \in F(x) \) and \( \min\{s, t\} \in F(y) \).

The two theorems that we will use to prove the existence of Nash networks when the payoff function satisfies the SCP (or SSP) are the following.

**Theorem 1** (adapted from Zhou 1994, [12]). Suppose \( A^n = \{0, \ldots, n-1\}^n \), \( F : A^n \rightarrow A^n \), \( a \mapsto F(a) \) is a correspondence from \( A^n \) to \( A^n \), and \( E \) is the set of fixed points of \( F \). If \( F \) is ascending in \( a \), then \( E \) is non empty.

**Theorem 2** (adapted from Kukushkin 1994, [7]). Let \( F_i : \prod_{j \in N \setminus \{i\}} A \rightarrow A \) be a correspondence allowing a decreasing single-valued selection. Then, the set of fixed points is nonempty.

**Proof of Proposition 2** We successively prove existence and 2.1, omitting the proof of 2.2 which is straightforward.
1. First, it is obvious that if the function $\phi$ satisfies SCP, then the best response correspondence is ascending. So, our model satisfies the assumption of Theorem 1 and a Nash network always exists.

Second, let the function $\phi$ satisfy SSP and let $BR_i(n_{-i}(g)) = \arg\max_{n_i(g) \in A} \{\phi(n_i(g), (n_{-i}(g))\}$. Then $\min_{n_i(g) \in A}\{BR_i(n_{-i}(g))\}$ is a decreasing single valued selection. Therefore, our model satisfies the assumption of Theorem 2 and a Nash network always exists.

2. Consider a non-empty and non-symmetric network $g$. We show that $g$ is not an equilibrium network. To create a contradiction, let $g$ be a Nash network. Since $g$ is a non-empty and non-symmetric network, there are player $i$ and player $j$ such that $N_i(g) \neq N_j(g)$. Without loss of generality, we assume that $n_i(g) \geq n_j(g)$. It follows that we have $n_{-i}(g) \leq n_{-j}(g)$. Moreover, since $g$ is a Nash network $\phi(n_i(g), n_{-i}(g)) - \phi(n_j, n_{-i}(g)) \geq 0$ since $n_i(g)$ is a best response for player $i$, and $\phi(n_j(g), n_{-j}(g)) - \phi(n_i(g), n_{-j}(g)) \geq 0$ since $n_j(g)$ is a best response for player $j$. But since $\phi$ satisfies SCP, we have: $\phi(n_i(g), n_{-j}(g)) - \phi(n_j(g), n_{-j}(g)) > \phi(n_i(g), n_{-i}(g)) - \phi(n_j(g), n_{-i}(g)) \geq 0$. It follows that $g$ is not a Nash network. This completes the proof of the first part of the proposition.

For the second part of the proposition, let $g$ be a symmetric network, with $n_i(g) = x$ for all $i$. It is sufficient to assume that $\phi$ satisfies SCP and $\phi(x, (n - 1)x)$ is the maximum of $\phi$.

\[ \square \]

6.3 Proof of proposition 4

**Proof of 4.1: Existence.** Suppose that payoff function satisfies equation (1), the strategic substitutes property and is convex. First, we prove that there always exists a Nash network and second, we characterize the Nash networks.

Let us define a process to show the existence of Nash networks. We start with the empty network, $g^0$. If it is not Nash, we order players in pre-specified
manner 1, 2, ..., n, and build an improving path with 3 steps. We denote by $g^j$ the network obtained at the end of step $j$, $j = 1, 2, 3$. We denote by $g^{i,j}$ the network formed when player $i$ is given the opportunity to play a best response in step $j$ and $n_k(g^{i,j})$ the number of links player $k$ has when player $i$ is given the opportunity to play a best response in step $j$.

**Step 1:** We give players 1, 2, ..., $n$ in turn the option to play a best response.

By symmetry of players, if player 1 has no incentive to form links in step 1, then no other player has any incentive to form links. However, since the empty network is not Nash, player 1 has an incentive to form links in $g^0$. It follows from convexity and SSP that each player $i = 1, ..., n - 1$ who plays a best response at Step 1 forms a number of links greater or equal to the number of links formed by player $i + 1$. Moreover, by the same properties, it is also straightforward that if player $i$ forms a link with player $j$, then player $i$ forms links with all player $j' > j$.

If all players are playing a best response in $g^1$, then we are done. Otherwise, we build a second step in the process of best responses: we give players $n - 1, ..., 1$ in turn the option to revise their choice. At the end of this step we get the network $g^2$.

**Step 2:** We give players $n - 1, ..., 1$ in turn the option to revise their choice.

We know that, for each player $i_0 = 1, ..., n$, we have $n_i(g^{i_0,1}) \leq n_i(g^{i_0,2})$ for all players $i \neq i_0$. Hence, by SSP, player $i_0$ does not have incentives to form new links in Step 2 and if player $i_0$ has an incentive to maintain her link with player $i$, then she has an incentive to maintain her links with all players $j > i$.

Moreover, by using convexity and SSP, it can be shown that if a player, say $i_0$, has an incentive to delete links with a set of players in Step 2, then no player $i > i_0$ has an incentive to form or maintain links with players belonging to this set in Step 2. Therefore, at the end of Step 2, in the network $g^2$, for all players $i$, we have: $g^{i,i+k}_2 = 1 \Rightarrow g^{i,i+k+1}_2 = ... = g^{i,n}_2 = 1$, $n_i(g^2) \leq n_i(g^1)$, and $n_i(g^2) \geq n_{i+1}(g^2)$.

In $g^2$ no player $i \in N$ has an incentive to delete links. In order to show that $g^2$ is a Nash network, we build a third step in the improving path and show that no player $i$ has an incentive to form new links in $g^2$.

**Step 3:** We give players $n, ..., 1$, in turn the option to revise their choice and play a best
response.

Suppose $i_0$ is the first player in step 3 who has an incentive to form new links. Let $j_0 < i_0$ be the first player in the set of players $\{1, 2, \ldots, i_0 - 1\}$ with whom $i_0$ has an incentive to form a new link. If player $i_0$ has an incentive to revise her choice and form new links in Step 3, then by SSP this implies that we have $n_i(g^2) < n_i(g^{i_0, 2})$ for at least one player $i \in Z = \{j_0, j_0 + 1, \ldots, i_0 - 1\}$. Let player $\ell_0 \in Z$ be the player such that $n_{\ell_0}(g^2) < n_{\ell_0}(g^{i_0, 2})$ and for all $i \in Z, i < \ell, n_i(g^2) = n_i(g^{i_0, 2})$. Now it is obvious by convexity and SSP that if player $i_0$ has an incentive to form these new links in Step 3, then player $\ell_0$ did not play a best response in $g^2$, giving us a contradiction.

**Proof of 4.1: Characterization.** In a non empty Nash network $g$, we show that for all $i \in N$ and $j \in N$, we have either $N_i(g) \subset N_j(g)$ or $N_i(g) \supset N_j(g)$. Suppose, without loss of generality, that player $i$ and player $j$ are such that $n_i(g) \geq n_j(g)$. Then, we denote by $k_i^M = \arg\max_{k \in N_i(g) \setminus N_j(g)}\{n_k(g)\}$ and by $k_j^m = \arg\max_{k \in N_j(g) \setminus N_i(g)}\{n_k(g)\}$. There are two possibilities either $k_i^M \geq k_j^m$, or $k_i^M < k_j^m$. We study successively the two cases. If $k_i^M \geq k_j^m$, then we have $\psi(n_i(g) + 1, k_j^m) > \psi(n_i(g), k_j^m) > \psi(n_i(g), k_i^M) > 0$. The first inequality comes from the convexity property, the second inequality comes from the SSP, and the third one results from the fact that $g$ is Nash. It follows that player $i$ has an incentive to form a link with player $k_j^m$, a contradiction. If $k_i^M < k_j^m$, then we have $\psi(n_j(g), k_j^m) < \psi(n_i(g), k_j^m) < \psi(n_i(g) + 1, k_j^m) < 0$. The first and the second inequality come from the convexity property and the third one follows the fact that $g$ is Nash. It follows that player $j$ has no incentive to form a link with player $k_j^m$, a contradiction. □

**Proof of 4.2:** Suppose that the payoff function satisfies SSP and is concave. We firstly prove the existence of Nash networks. We begin with the empty network. Either the empty network is Nash and we are done, or there is a player, say 1, who has an incentive to form a link with player $n$. It follows that $\psi(1, 0) > 0$. We let player 1 form a link with player $n$ and we obtain the network $g^1$. Then, we let all players $j \in N \setminus \{1, n\}$ form a link with player $n$. We obtain a network $g^2$. We let now player $n$ choose to add a link with player 1 or not. If she does not add this link, then $\psi(1, 1) < 0$ and no player has an incentive
to add a link since $\psi(k, 1) < \psi(1, 1) < 0$, $k \geq 2$, by concavity. Therefore, the network $g^2$ is Nash and we are done. If player $n$ has an incentive to form a link with player 1, then $\psi(1, 1) > 0$. It follows that no player $j \in N \setminus \{n\}$ has an incentive to delete the link she has formed with player $n$. We obtain a network $g^3$. We reiterate this process with players $n - 1, n - 2, \ldots$, instead of player $n$ and we show that there is a Nash network since the number of links that each player can form is finite.

We now show that in a Nash network $g$ we have $|N(g)| \leq 2$. Suppose $|N(g)| > 2$, then there are three players, say $i, j$ and $k$ such that $n_i(g) < n_j(g) < n_k(g)$. We show that player $j$ does not play a best response. We have $n_k(g) \geq n_i(g) + 2$. It follows that there is a player $\ell$ such that player $k$ has formed a link with $\ell$ and player $i$ has not formed a link with $\ell$ in $g$. Since player $k$ and player $i$ play a best response in $g$, we have $\psi(n_k(g), n_\ell(g)) > 0$ and $\psi(n_i(g) + 1, n_\ell(g)) < 0$. Moreover, by concavity, we have $\psi(n_i(g) + 1, n_\ell(g)) > \psi(n_k(g), n_\ell(g))$, a contradiction. □

References


