Both in fuzzy noncooperative games and in abstract economies, players impose mutual restrictions on their available strategic choices. In the existing theory of fuzzy games, a player is only concerned about how to minimize the restrictions he imposes on others, does not explicitly pursue any other objectives and ignores the restrictions others may impose upon him. In the prevailing formal treatment of abstract economies, a player respects the restrictions imposed by others upon him, disregards any restrictions that his actions may inflict upon others and follows separate objectives of his own. Here we combine two aspects of the two theories: A player tries to minimize the restrictions he imposes on others and at the same time respects the restrictions imposed by others upon him, but does not explicitly pursue any other objectives of his own. We establish existence of an equilibrium in this framework.
1 Introduction

This paper provides a new formulation of noncooperative fuzzy games. Such games were originally developed by Butnariu (1978, 1979) and later revised by Billot (1992). A fuzzy game is modeled using the notion of a fuzzy set introduced in a seminal paper by Zadeh (1965). A fuzzy set differs from a classical set (from now on referred to as a crisp set) in that the characteristic function can take any value in the interval $[0,1]$. In the Butnariu-Billot formulation each player’s beliefs about the actions of the other players are modeled as fuzzy sets. These beliefs need to satisfy an axiom that constrains the actions available to others. The equilibrium relies on a restrictive assumption involving only those situations where players’ beliefs are commonly known. Since these beliefs are now perfect information, they do not constrain the actions of the other players with any degree of uncertainty. Without this assumption the above model need not have an equilibrium, though possible solutions to the game may still exist.

The existing verbal and formal descriptions of fuzzy noncooperative games by Butnariu (1978, 1979) and Billot (1992) seem somewhat enigmatic and perhaps “fuzzy” to the unfamiliar reader. In this paper we first recast the Butnariu-Billot model using standard game-theoretic and crisp set terminology. We then relax the Butnariu-Billot axiom by requiring that player $i$’s beliefs should allow the other players to choose any mixed strategy from their available choices. Further, in equilibrium instead of requiring full information about each others’ beliefs, we only impose mutual consistency of beliefs. Despite these two modifications, the equilibrium concept is still quite weak. We demonstrate this through the example of an abstract economy. In an abstract economy each player maximizes his own objectives subject to the constraints imposed on his actions by the others. We develop a new model of fuzzy noncooperative games by marrying the two types of possible restrictions on the actions of the players, one derived from the Butnariu-Billot formulation and one from the model of abstract economies. In our model, each player tries to minimize the restrictions he imposes on others while respecting the constraints imposed on his own actions by the others, but does not explicitly pursue any objectives of his own. This allows us to ensure the existence of an equilibrium in the reformulated fuzzy game.

The remaining sections of the paper are organized as follows. In Section 2 mathematical tools required in the rest of the paper are presented. These
cover concepts from fuzzy set theory as well as abstract economies. Section 3 contains the crisp version of the Butnariu-Billot model as well as our refor-
mulation. Section 4 concludes and the Appendix provides a brief summary of Butnariu’s formulation.

2 Mathematical Preliminaries

In this section we set forth the basic mathematical definitions that will be used in later sections of the paper. We will first introduce the notion of fuzzy sets. This is followed by the introduction of relevant material on abstract economies that are necessary for our re-formulation.

2.1 Relevant Concepts From Fuzzy Set Theory

The earliest formulation of the concepts of fuzzy sets is due to Zadeh (1965) who tried to generalize the idea of a classical set by extending the range of its characteristic function. Informally, a fuzzy set is a class of objects for which there is no sharp boundary between those objects that belong to the class and those that do not. Here we provide some definitions that are pertinent to our work.

Let \( X \) denote the universe of discourse. We distinguish between “crisp” or traditional and fuzzy subsets of \( X \).

**Definition 1** The characteristic function \( \Psi_A \) of a crisp set \( A \) maps the elements of \( X \) to the elements of the set \( \{0, 1\} \), i.e., \( \Psi_A : X \rightarrow \{0, 1\} \). For each \( x \in X \),

\[
\Psi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{otherwise}
\end{cases}
\]

To go from here to a fuzzy set we need to expand the set \( \{0, 1\} \) to the set \( [0, 1] \) with 0 and 1 representing the lowest and highest grades of membership respectively.

**Definition 2** The membership function \( \mu_A \) of a fuzzy set \( A \) maps the elements of \( X \) to the elements of the set \( [0, 1] \), i.e., \( \mu_A : X \rightarrow [0, 1] \). For \( x \in X \), \( \mu_A(x) \) is called the degree or grade of membership.
Membership functions have also been used as belief functions and can be viewed as non-additive probabilities. For a discussion of these issues see Klir and Yuan (1995) and Billot (1991). The fuzzy set $A$ itself is defined as the graph of $\mu_A$:

$$A = \{(x, y) \in X \times [0, 1] : y = \mu_A(x)\}.$$  

The only purpose of this definition is to have something at hand that is literally a set. All the properties of fuzzy sets are defined in terms of their membership functions. For example, the fuzzy set $A$ is called normal when $\sup_x \mu_A(x) = 1$. To emphasize that, indeed, all the properties of fuzzy sets are actually attributes of their membership functions, suppose that $X$ is a nonempty subset of a Euclidean space. Then $A$ is called convex, if $\mu_A$ is quasi-concave. This does not mean, however, that the graph of $\mu_A$ is convex. Take in particular a crisp set $A$. Then $A$ as a subset of $X$ is convex if and only if its characteristic function $\Psi_A$ is quasi-concave. The latter does not imply, however, that the graph of $\Psi_A$ is convex. We highlight two further important definitions, again in terms of membership functions.

**Definition 3** The fuzzy set $B$ is a subset of the fuzzy set $A$ if and only if

$$\mu_B(x) \leq \mu_A(x)$$

for all $x \in X$.

For an axiomatic discussion of the standard set operations like union, intersection, etc. in the context of fuzzy sets see Bellman and Giertz (1973). The upper contour sets of a fuzzy set are called $\alpha$-cuts and introduced next.

**Definition 4** Let $\alpha \in [0, 1]$. The crisp set $A_\alpha$ of elements of $X$ that belong to the fuzzy set $A$ at least to the degree $\alpha$ is called the $\alpha$-cut of the fuzzy set $A$:

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$$

Moreover, we define the strict $\alpha$-cut $A^*_\alpha$ of $A$ as the crisp set

$$A^*_\alpha = \{x \in X : \mu_A(x) > \alpha\}.$$  

In particular, $A_0 = X$ and $A_1^* = \emptyset$. $A_0^*$ is called the support of $A$ or $\mu_A$.

Detailed expositions of all aspects of fuzzy set theory and their numerous applications can be found in the textbooks by Zimmermann (1990) and Klir and Yuan (1995).
2.2 Relevant Concepts From Abstract Economies

In this section we introduce some basic elements of abstract economies. The standard model of strategic games assumes in fact that each player is free to choose whatever action he pleases from his strategy set, regardless of the actions of others. The objectives of the player are represented by a utility or payoff function defined on the set of joint strategies or action profiles which add a cardinal flavor to the model. We shall turn to abstract economies which allow for (a) dependence of a player’s feasible actions on the choices made by other players and (b) an ordinal concept of (not necessarily transitive and complete) preferences. This setting lends itself quite naturally to fuzzification.

2.2.1 Preferences and Constraints in Abstract Economies

For Nash equilibria in pure strategies of strategic or normal form games only the ordinal preferences of players matter. But one resorts frequently to Nash equilibria in mixed strategies and then the cardinal aspects of the payoff functions become essential. The situation is quite different in the context of abstract economies, also known as “generalized games” or “pseudo-games”. An abstract economy assumes the form

$$\Gamma = (I; (S_i)_{i \in I}; (P_i)_{i \in I}; (F_i)_{i \in I})$$

where

1. $I$ is a non-empty set of players;
2. $S_i$ is a non-empty strategy set (strategy space), representing the strategies $s_i$ for player $i \in I$;
3. $P_i : S \rightarrow S$ is a strict preference relation on $S \equiv \times_{j \in I} S_j$ for each player $i \in I$;
4. $F_i : S \rightarrow S_i$ is the constraint relation for each player $i \in I$.

$F_i$ tells which strategies are actually feasible for player $i$, given the strategy choices of the other players. For technical convenience, we have written $F_i$
as a function of the strategies of all the players including player \( i \). In most applications, \( F_i \) is independent of \( i \)'s choice. For instance, \( i \) cannot take a chair taken by somebody else. In an economic context, a fictitious player known as the auctioneer may set prices and thus determine the budget sets of other players. The jointly feasible strategies are the fixed points of the relation \( F = \times_{j \in I} F_j : S \mapsto S \). In principle, \( F_i(s) \) can be empty for some \( i \in I \) and \( s \in S \). However, if this happens too often and \( F \) does not have a fixed point, then the theory becomes vacuous. If at the other extreme, \( F_i(s) = S_i \) for all \( i \) and \( s \), then \( \Gamma \) is an ordinal game. Following Border (1985), let us define, for each \( i \in I \), the good reply relation \( U_i : S \mapsto S_i \) by \( U_i(s) \equiv \{ s' \in S_i : (s', s_{-i}) \in P_i(s) \} \) for \( s = (s_i, s_{-i}) \in S \). An equilibrium of the abstract economy \( \Gamma \) is a strategy profile \( s \in S \) which is jointly feasible (a fixed point of \( F \), i.e. \( s \in F(s) \)), and does not permit a feasible good reply, i.e. \( U_i(s) \cap F_i(s) = \emptyset \) for all \( i \in I \). The following existence result which is also stated and demonstrated in Border (1985) is of particular interest to us, since it does not require transitivity or completeness of preferences.

**Theorem 1 (Shafer and Sonnenschein (1975))** Suppose that for each \( i \),

1. \( S_i \) is a nonempty, compact and convex subset of a Euclidean space;
2. \( F_i \) is continuous and has nonempty, compact and convex values;
3. \( U_i \) has open graph in \( S \times S_i \);
4. \( s_i \) does not belong to the convex hull of \( U_i(s) \) for all \( s \in S \).

Then the abstract economy \( \Gamma \) has an equilibrium.

### 2.2.2 Fuzzification of Preferences and Constraints

Binary relations from a set \( Y \) to a set \( Z \) are easily fuzzified. Namely, a binary relation \( R \) from \( Y \) to \( Z \) can be identified with its graph, \( Gr_R = \{ (y, z) \in Y \times Z : z \in R(y) \} \), a subset of \( X = Y \times Z \). In that sense, the binary relations from \( Y \) to \( Z \) are the crisp subsets of \( X \). Accordingly, the fuzzy binary relations from \( Y \) to \( Z \) are the fuzzy subsets of \( X \). Fuzzy preferences and choice based on such preferences have been explored among
others by Basu (1984), Barret et al. (1990) Sengupta (1999), and Pattanaik and Sengupta (2000). Basu (1984) fuzzifies revealed preference theory, where fuzzy preferences lead to exact choices. It is shown that the a choice rule $C(.)$ which can be rationalized by the fuzzy preferences exists. The paper also provides comparisons with the traditional theory. Barret et al. (1990) argue that while people in general have vague preferences they make exact choices. They investigate plausible rationality properties for two different types of choice rules. The first is called a binary choice rule under which the choice from any set is basically derived from choices made from two-element. Non-binary choice rules where this assumption is relaxed are also explored in this paper. Sengupta (1999) considers agents with fuzzy preferences making exact choices. He provides an axiomatic characterization of a class of binary choice rules called $\alpha$—rules. According to this rule, for any $\alpha \in [0, 1]$, an alternative $x$ is chosen over $y$ when in any pairwise comparison involving the two, $x$ is preferred over $y$ by at least a degree of $\alpha$. Pattanaik and Sengupta also consider a situation where an agent with fuzzy preferences makes exact choices. They confine attention only to feasible sets containing no more than two alternatives and provide an axiomatic characterization of two broad classes of decision rules called ratio rules and difference rules. For a given fuzzy preference relation $R$, an alternative $x$ is chosen over $y$ according to the ratio rule if there exists $\alpha_{\{x,y\}} \in [0, 1]$ such that $R(x, y) \geq \alpha_{\{x,y\}} R(y, x)$. An alternative ratio rule can be defined by using the strict inequality. A difference rule on the other hand requires that the difference between $R(x, y)$ and $R(y, x)$ be bounded by $\varepsilon_{\{x,y\}} \in [0, 1]$. The appeal of this theory clearly lies in the fact that it allows agents to be somewhat fuzzy in their ranking of alternatives, thus embodying different degrees of rationality and yet making exact choices.

Returning to abstract economies $\Gamma$, we can replace both the preference relations $P_i$ or $U_i$, respectively, and the constraint relations $F_i$ by fuzzy versions. The above existence theorem readily applies to the various crisp relatives of these relations:

- If we merely require that a relation holds with a nonzero degree, we can work with the support of the relation.
- If we require that the relation holds with at least a given degree $\alpha$, we can work with the corresponding upper contour set ($\alpha$-cut) of the relation.
Notice that a higher \( \alpha \) for the \( F_i \) makes joint feasibility harder whereas a higher \( \alpha \) for the \( F_i \) and \( U_i \) furthers the absence of feasible good replies. The first effect may eliminate some equilibria. The second effect may create new equilibria.

3 Fuzzy Games: A Reformulation

In this section we develop a stripped down and crisp version of the standard Butnariu-Billot model. The essential idea is first — and we think better — presented in crisp terms. Fuzzy elements will be introduced later. We begin with the discussion of a certain axiom, labelled Axiom 1, which in our context constitutes the counterpart of Axiom A of the Butnariu-Billot model reported in the Appendix. We put forward an argument that demonstrates the frequent invalidity of Axiom 1. We next discuss the merits of a new and weaker Axiom 2 which is still very restrictive, but not to the extent of being a priori invalid. Finally, we develop a new model of fuzzy noncooperative games which does not rely on any of these axioms and can be cast within the framework of abstract economies.

3.1 Preliminary Formulation

Consider an underlying game form \( GF = (I; (S_i)_{i \in I}) \). A game form is a strategic form without a specification of the payoff functions. For simplicity we assume a finite game form, in particular, \( I = \{1, \ldots, n\} \). For each player \( i \in I \), let \( Y_i = \Delta(S_i) \) denote the set of mixed strategies. Let \( Y = \times_{j \in I} Y_j \) and let \( Y_{-i} = \times_{j \neq i} Y_j \). Each player \( i \) has in addition individual perceptions of which mixed strategy profiles \( y \in Y \) are feasible. The perceptions depend on player \( i \)'s reasoning process as well as her notion of how the other players would reason in the game. These perceptions are represented by a subset \( \pi_i \) of \( Y \). In player \( i \)'s view if she chooses \( y_i \in Y_i \), then only elements \( y_{-i} \) in \( \pi_i(y_i) \), the \( y_i \)-section of \( \pi_i \) are feasible for the other players. Formally,

\[
\pi_i(y_i) = \{ y_{-i} \in Y_{-i} : (y_i, y_{-i}) \in \pi_i \}.
\]

Finally, player \( i \) has preferences over subsets of \( Y_{-i} \) induced by set inclusion:

\[
A_{-i} \subseteq B_{-i} \iff A_{-i} \subseteq B_{-i} \quad \text{for} \quad A_{-i}, B_{-i} \subseteq Y_{-i}.
\]
We can now define an equilibrium.

**Definition 5** An equilibrium is a profile \( y^* = (y^*_1, \ldots, y^*_n) \in Y \), such that the following two conditions hold.

(a) Mutual consistency: for all \( i \), \( y^*_{-i} \in \pi_i(y^*_i) \).

(b) Preference maximization: for all \( i \), there is no \( y_i \in Y_i \) such that \( \pi_i(y_i) \geq \pi_i(y^*_i) \).

The mutual consistency requirement is a condition on the player’s perceptions which requires that in equilibrium, each player’s beliefs about the others include the equilibrium strategy profile. Condition (b) means that a player wishes that her own choice restricts the choices available to the others as little as possible.

Now consider the following axiom which is the crisp version of Axiom A suggested in the literature; see Appendix.

**Axiom 1**: For each \( i \in I \) and \( A_{-i} \subseteq Y_{-i} \) with \( A_{-i} \neq \emptyset \), there exists \( y_i \in Y_i \) such that \( A_{-i} = \pi_i(y_i) \).

Note that this axiom is violated, unless all \( Y_{-i} \) are singletons. For suppose \( Y_{-i} \) is not a singleton. Then \( Y_{-i} \) has the cardinality \( c \) of the set of the real numbers. Hence \( P(Y_{-i}) \), the power set of \( Y_{-i} \) has cardinality \( 2^c > c \). So has \( P(Y_{-i}) \setminus \emptyset \). On the other hand, \( Y_i \) has a cardinality of at most \( c \). Therefore the mapping \( y_i \mapsto \pi_i(y_i) \) cannot have an image that contains \( P(Y_{-i}) \setminus \emptyset \).

Fortunately Axiom 1 can be replaced by a weaker one.

**Axiom 2**: For each \( i \in I \), there exists \( y_i \in Y_i \) such that \( Y_{-i} = \pi_i(y_i) \).

Using this less demanding axiom, we can state the following result.

**Proposition 1** Suppose Axiom 2 holds. Then an equilibrium exists.

**Proof**: By Axiom 2, we can choose for each \( i \in I \), a \( y^*_i \in Y_i \) such that \( Y_{-i} = \pi_i(y^*_i) \). Let us choose such a \( y^*_i \). Then \( y^*_{-i} \in Y_{-i} = \pi_i(y^*_i) \) for all \( i \). Hence Condition (a) is satisfied. Moreover, for all \( i \in I \) and \( y_i \in Y_i \), \( \pi_i(y_i) \subseteq \pi_i(y^*_i) \). Hence (b) holds as well.

\footnote{Observe that the same reasoning applies to Axiom A in the Appendix and any fuzzy version of Axiom 1.}
The appeal of the equilibrium is Condition (b) which lets a player maximize based only on his subjective perception of the others and is not affected by their actual play, akin to the solvability concept of von Neumann and Morgenstern for two-person zero-sum games and of Moulin for dominance solvable games.

The existence result, however, still hinges on the very restrictive Axiom 2. This can be easily demonstrated through a simple example.

**Example 1** Consider the case where \( I = \{1, 2\} \) and \( |S_1| = |S_2| = 2 \). Then we can set \( Y_i = [0, 1] \) for each \( i \in I \) where \( y_i \in Y_i \) stands for the probability that \( i \)'s “first action” is played. Let \( \pi_1 \) be given by \( \pi_1(y_1) = \{ y_2 : 1 - y_1/2 \leq y_2 \leq 1 \} \) and \( \pi_2 \) by \( \pi_2(y_2) = \{ y_1 : 0 \leq y_1 \leq 1 - y_2/2 \} \). Then Axiom 2 is violated. Further \( y^* = (0, 1) \) is the only point in \( Y \) that satisfies condition (a) and \( y^* = (1, 0) \) is the only point in \( Y \) that satisfies condition (b). Thus no equilibrium exists. The example is depicted in Figure 1.

Hence, we face the dilemma of either making the strong assumption of Axiom 2 which renders existence almost trivial, or possibly lacking existence of an equilibrium.
3.2 The Reformulation

We suggest a way out by moving away from the time-honored von Neumann-Morgenstern approach and closer to the contemporary theory of strategic games. We propose that a player should be aware of the constraints that the choices of others impose on his own play in addition to the perceived restrictions that his play imposes on the choices of others. To illustrate the idea, consider again the parameters used in the previous example: \( I = \{ 1, 2 \} \), \(|S_1| = |S_2| = 2 \) and \( Y_1 = Y_2 = [0, 1] \). We assume that all constraints are interval constraints, i.e., \( \pi_i(y_i) \subseteq Y_{-i} \) is a non-empty interval \([a_i(y_i), b_i(y_i)]\). We also assume that the two functions \( a_i : Y_i \to Y_{-i} \) and \( b_i : Y_i \to Y_{-i} \) determining \( \pi_i \) are continuous. Then \( \pi_i \) is connected and path-connected. Let us assume, moreover, that \( \pi_i \subseteq Y \) is convex. This is equivalent to \( a_i \) being convex and \( b_i \) being concave and implies that \( l_i = b_i - a_i \) is concave.

Now consider the abstract economy \( \Gamma = (I; Y_1, Y_2; P_1, P_2; \pi_1, \pi_2) \) with player set \( I = \{ 1, 2 \} \), strategy sets \( Y_1 \) and \( Y_2 \), and constraint relations \( \pi_1 \) and \( \pi_2 \) together with preference relations \( P_1 \) and \( P_2 \) defined as follows. \( P_i \) is the strict preference obtained when \( l_i : Y_i \to \mathbb{R} \) is interpreted as a payoff function. Then conditions (i)-(iv) of the Shafer-Sonnenschein theorem are met. Hence we have the following proposition.

**Proposition 2** \( \Gamma \) has an equilibrium.

Notice that \( l_i(y_i) \) is the length of the interval \( \pi_i(y_i) \). Hence if we replace the \( P_i \) by the preference relation \( \succ_i \) induced by set inclusion we have \( y_i' \succ_i y_i \) implies \( y_i' \succ_i y_i \). This gives us the following corollary.

**Corollary 1** The abstract economy \( \Gamma^0 = (I; Y_1, Y_2; \succ_1, \succ_2; \pi_1, \pi_2) \) has an equilibrium.

**Proof:** Let \( (y_1^*, y_2^*) \) be an equilibrium of \( \Gamma \). We claim that \( (y_1^*, y_2^*) \) is an equilibrium of the abstract economy \( \Gamma^0 \). \( (y_1^*, y_2^*) \) is socially feasible. It remains to show that none of the players has a feasible good reply. Suppose player \( i \) has one in \( \Gamma^0 \). Then player \( i \) also has one in \( \Gamma \) which yields a contradiction, since \( (y_1^*, y_2^*) \) was assumed to be an equilibrium of \( \Gamma \). \( \square \)
Comparing feasible sets by their size (Lebesgue measure) is not merely a technical trick, but seems to us an appealing alternative to comparison via set inclusion. The proposition and its corollary ensure that either type of preferences can be accommodated. Incidentally, after the reformulation, the abstract economies $\Gamma$ and $\Gamma^0$ associated with Example 1 have both $(0, 1)$ as the only equilibrium.

This new model of fuzzy games can be generalized to more than two players, more than two actions per player and fuzzy $\pi_i$. How the latter can be achieved has been outlined in the previous section.

4 Conclusion

The modified version of noncooperative fuzzy games developed here is a merger of two different ideas. By allowing players to minimize the restrictions they impose on others we allow for a larger choice set and in a certain sense a wider set of rational behaviors. Moreover, since we do not require perfect information about beliefs, the equilibrium permits a richer set of possibilities. In the earlier version of fuzzy games, the rationality issue and the models that players have of each others behavior was a moot point owing to the full information requirement. By incorporating the notion of respecting the constraints imposed by others from the abstract economies literature, we restrict the players to choosing feasible actions. This feature enables us to show the existence of equilibrium without requiring complete knowledge of beliefs. Note that like in the earlier literature on fuzzy games, our model still does not involve maximizing an explicit objective.

Finally, it is also worth pointing out that the adopted approach is somewhat akin to Bellman and Zadeh’s (1972) work on decision theory in a fuzzy environment. The decision maker’s choices are constrained by two types of beliefs in their framework. The first is a goal function which allows an agent to rank the inexact outcomes associated with the different choices, and the second is a constraint function which allows the agent to rank the set of feasible alternatives. The optimal action is determined by the intersection of these two sets. A major difference with our approach is the fact that besides the game-theoretic setting, we have no explicit objectives. Further, in our model the two types of constraints are well defined and quite distinct, while in Bellman and Zadeh’s own words in their model “… the goals and/or
the constraints constitute classes of alternatives whose boundaries are not sharply defined."
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Appendix: Fuzzy Games à la Butnariu and Billot

In this section we provide a brief review of the existing work on fuzzy strategic or noncooperative games. They were first developed by Butnariu (1978, 1979) and later refined by Billot (1992). Billot’s main contribution in this area has been to provide a better interpretation of Butnariu’s work making it more accessible to the reader. However, as will be evident the notation used in this formulation is still quite cumbersome and many standard concepts have been labelled differently. Our main objective here is to interpret their work using standard terminology and indicate the weaknesses of the formulation.

Butnariu defines a \( n \)-person noncooperative fuzzy game in normal form as \( \Gamma = (S_i, Y_i, \Pi_i)_{i=1}^n \), where the set of players is denoted by \( I = \{1, \ldots, n\} \), such that for any player \( i \in I \) the following four conditions are satisfied:

1. Each player’s set of pure strategies is given by \( S_i \),

2. We define an element of \( Y_i \) as \( w^i = (w^i_1, w^i_2, \ldots, w^i_m) \), where \( w^i_m \) denotes the weight assigned to player \( i \)'s to her \( m \)-th pure strategy. Each \( w^i \in Y_i \) is called a strategic arrangement of player \( i \in I \). The \( n \)-dimensional vector \( w = (w^1, \ldots, w^n) \in Y = \times_{i \in I} Y_i \) is called a strategic choice in \( \Gamma \).

3. \( \Pi_i \in 2^Y \) and for all \( w \in Y, \pi_i(w) \) is the possibility level assigned by player \( i \) to the strategic choice \( w \). This possibility level is essentially a membership function and denotes the membership value of each mixed strategy profile as assessed by player \( i \).

4. Let \( Y_{-i} = \times_{j \neq i} Y_j \) and let \( W_i = 2^{Y_{-i}} \times Y_i \). Then \( s_i = (A^i_j, w^i) \in W_i \) is player \( i \)'s strategic conception in \( \Gamma \).

Also the following axiom is assumed to hold:

**Axiom A**: If \( A^i_j \in 2^{Y_{-i}} \) and \( A^i_j \neq \emptyset \), then \( \pi_i(A^i_j) \neq \emptyset \), i.e., there exists \( s_i \in W_i \) such that \( \pi_i(A^i_j)(w^i) \neq \emptyset \).

The second condition is just an alternative way of defining mixed strate-
gies, where the players are assumed to know the weights of the mixed strategies. Of course this leads to a certain amount of redundancy, since one could just assume the players know the weights on the mixed strategies, which would imply automatic knowledge of the pure strategies. Alternatively we could assume that they just know the pure strategies, with the set of mixed strategies being all possible probability distributions over the pure strategies. Note that $\pi_i \in 2^Y_i$, implying that the beliefs about mixed strategy profiles are actually crisp sets. The strategic conception itself consists of player $i$'s beliefs about the other players and his own mixed strategy. Hence the definition of mixed strategies using probability weights in this formulation has an advantage in the sense that the two components of the strategic conception now lie in the interval $[0, 1]$. The axiom states that for a nonempty set of beliefs about the strategies of the other players, player $i$ can choose a mixed strategy in response in the game which will constitute a strategic conception. In other words, there exists a $w^i$ such that $A^i_f$ is the $w^i$-section of $\pi_i$. Also let $W = \times_{i \in I} W_i$.

**Definition 6** Let $\Gamma$ be an $n$-person noncooperative game satisfying the 4 conditions and the axiom stated above. A **play** is a vector $s = (s_1, \ldots, s_n) \in W$.

**Definition 7** Let $s^*_i$ and $\bar{s}_i$ denote two strategic conceptions of player $i$. We say that $s^*_i$ is a **better strategic conception** than $\bar{s}_i$, or $s^*_i \succ_i \bar{s}_i$ for player $i$ if and only if $\pi_i(A^*_i(w^i)) > \pi_i(A^i_f)(\bar{w}^i)$.

In other words, we say that $s^*_i \succ_i \bar{s}_i$ for player $i$ if and only if $A^*_i \subset A^i_f$.

**Definition 8** Let $s^*$ and $\bar{s}$ denote two different plays of the game. We say that $s^*$ is **socially preferred** to $\bar{s}$, or $s^* \succ \bar{s}$ if and only if for all $i \in I$, $\pi_i(A^*_f)(w^i) > \pi_i(A^i_f)(\bar{w}^i)$.

Hence, $s^* \succ \bar{s}$ if and only if $s^*_i \succ_i \bar{s}_i$ for all $i \in I$.

**Definition 9** **A possible solution** of the game $\Gamma$ is a play $s^*$, such that for any other play $\bar{s}$, where for all $i \in I$, $s^*_i = (A^*_f, w^i)$, the play $\bar{s}$ cannot be socially preferable to $s^*_i$ if for all $i \in I$, and for all $\bar{w}^i \in Y_i$, we have $\pi_i(A^*_f)(w^i) \geq \pi_i(A^i_f)(\bar{w}^i)$. 

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The possible solution requires that the \( w^i - \) section of \( \pi_i \) corresponding to the equilibrium belief \( A^i_f \) is greater for \( w^{i*} \) than for \( \tilde{w}^i \). Intuitively, the possible solution can be interpreted as two conditions. The first condition says \( s^* \) is a possible solution if \( s^*_i \) is feasible, that is \( A^i_f \) is the \( w^i - \) section of \( \pi_i \). This implies that Axiom A is satisfied. The second condition requires that there is no \( \tilde{s} \) such that (a) \( \tilde{s}_i \) is feasible for all \( i \in I \), and (b) \( \tilde{s} \not\succ s^* \).

In order to define an equilibrium Butnariu allows for communication among the players. This communication allows players to reveal their beliefs to each other, while allowing them complete freedom in their choice of strategy. Given that \( \Pi_i \) for all \( i \in I \) is already a part of the definition of the game, this can only mean that players reveal their specific \( \pi_i \) to each other. Based on this we only consider what Butnariu calls plays with perfect information which is defined as follows:

**Definition 10** A play \( s^* = (A^i_f, w^{i*}) \) is called a play with perfect information when it is of the form

\[
A^i_f(w^1, \ldots, w^{i-1}, w^{i+1}, \ldots, w^n) = 1, \text{ for } w^j = w^{j*} \text{ where } j \neq i \\
= 0 \quad \text{otherwise.}
\]

We can alternatively replace this with the requirement that players have mutually consistent beliefs, or \( w^{-i*} \in A^i_f \) for all \( i \in I \). It should also be immediately obvious that such a play makes the game and the equilibrium concept which only allows for perfect information, quite uninteresting. Using this we can now define an equilibrium of the fuzzy game \( \Gamma \).

**Definition 11** An equilibrium point of the game \( \Gamma \) is a possible solution \( s^* \), where \( s^*_i = (A^i_f, w^{i*}) \) which satisfies the mutual consistency condition on beliefs for all players \( i \in I \).

Two existence proofs are also provided in this literature. The first theorem proves the existence of possible solutions and the second one proves the existence of equilibrium points in \( \Gamma \). However, in view of our earlier comments about the nature of the equilibrium, details of these proofs are omitted. The interested reader may refer to Butnariu (1979) and Billot (1992). A smorgasbord of fuzzy fixed point theorems can be found in Butnariu (1982).